

All photons are equal but some photons are more equal than others

Falk Töppel,^{1,2,*} Andrea Aiello,^{1,2} and Gerd Leuchs^{1,2}

¹Max Planck Institute for the Science of Light, Günther-Scharowsky-Straße 1/Bau 24, 91058 Erlangen, Germany

²Institute for Optics, Information and Photonics,
Universität Erlangen-Nürnberg, Staudtstr. 7/B2, 91058 Erlangen, Germany
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Photons indistinguishability is studied from a pragmatic, rather than speculative, point of view. As consequence of a detailed analysis of the coincidence probability distribution in a Hong-Ou-Mandel experiment, we introduce the concept of rate of distinguishability of photons. For photons prepared in a given, but arbitrary, state, this parameter allows one to make a quantitative analysis of their indistinguishability. Among other things, our study reveals that coupling between different degrees of freedom of the electromagnetic field, plays a crucial role in photons indistinguishability.

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Introduction.—Although particles indistinguishability has been a tenet of quantum mechanics long since, it is still a not well understood concept [1]. Actually, there is a large amount of literature on this subject, mostly laying in the realm of “foundations of quantum mechanics”, frequently nearby the edge between physics and philosophy [2–5]. However, besides being a central *conceptual* issue of modern physics, particles indistinguishability is also the key to *experimental* tests of fundamental laws of nature, as the spin-statistic theorem which is a pillar of the standard model of particles and forces [6, 7]. In quantum electrodynamics, particles (photons) indistinguishability manifest itself when two equally prepared photons interfere at the two input ports of a 50/50 beam splitter (BS). The joint probability of detection (coincidence probability) at the two outputs is exactly zero. Vice versa, when the two photons are prepared in a different manner, the coincidence probability raises up to 50%. This effect was first demonstrated by Hong *et al.* [8] and rapidly became central in a broad range of experiments in quantum physics [9–12]. Very recently, photon indistinguishability proved to be of special interest also for quantum information processing [13, 14]. Despite such a lively research activity, at the present time some misconceptions still remain regarding indistinguishable particles. For example, it is a common idea that “[...] quantum mechanical particles are too small for one to attach physical labels; they don’t have enough degrees of freedom to enable one to mark each particle differently” [15].

In this Letter we demonstrate that, contrary to popular belief, particles (specifically: photons) *do have* enough degrees of freedom to permit one to assign different distinguishing marks to each particle, as many as the number of degrees of freedom (DOFs) of the photons themselves. To this end, we introduce in an operational manner the *rate of distinguishability* of photons which appears to be a very useful “tool”. Our study shows that coupling between different DOFs critically *affects* photons indistinguishability. These results suggest the need for replacement of the strong concept of “particles in-

distinguishability” tout court, with the weaker concept of “particles indistinguishability with respect to a given degree of freedom”.

Two-photon interference.—To begin with, let us consider the two-photon interference experiment sketched in Fig. 1. [8]. The two photons impinging upon the 50/50

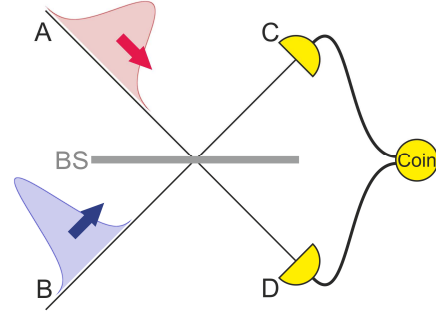


FIG. 1. (color online) Two-photon interference at a 50/50 beam splitter. Two independently prepared photons enter the two input ports *A* and *B* of the BS. They are eventually detected by two distinct detectors placed behind the output ports *C* and *D* of the BS. The plane of the figure is the *plane of incidence*

BS are prepared in the product state $|\psi^{AB}\rangle = |\psi^A\rangle|\psi^B\rangle$, where $|\psi^A\rangle = \hat{a}^\dagger[\psi^A]|0\rangle$ ($|\psi^B\rangle = \hat{b}^\dagger[\psi^B]|0\rangle$) denotes the single-photon state in arm *A* (*B*), with $\hat{a}^\dagger[\psi^A]$ ($\hat{b}^\dagger[\psi^B]$) being the operator that creates one photon in the wave-packet mode ψ^A (ψ^B) [16, 17]. The *normalized* function $\psi^{A,B}$ completely fixes the spectral, polarization and spatiotemporal characteristics of the photon entering port *A*, *B*. Annihilation and creation operators associated to orthogonal wave-packet modes, do commute: $[\hat{a}[\psi], \hat{b}^\dagger[\phi]] = (\psi, \phi) \delta_{ab}$, where (ψ, ϕ) denotes the scalar product in the complex linear space $\mathcal{L} \ni \psi, \phi$ [18]. The probability of detecting the two photons at the two output ports *C* and *D* is given by [19]:

$$P_{1,1}[\psi^A, \psi^B] = (1 - |\langle\psi^A|\psi^B\rangle|^2) / 2, \quad (1)$$

where $\langle \psi^A | \psi^B \rangle = (\psi^A, \psi^B)$.

Now, assume that the two single-photon wave packet modes ψ^A and ψ^B are prepared “almost” in the same manner, namely $\psi^A = \psi$ and $\psi^B = (\psi + \delta\psi) / \|\psi + \delta\psi\| \equiv \widetilde{\psi + \delta\psi}$. The functional variation of the coincidence probability generated by $\delta\psi$ will be, by definition: $\Delta P_{1,1}[\psi] \equiv P_{1,1}[\psi, \widetilde{\psi + \delta\psi}] - P_{1,1}[\psi, \psi] = P_{1,1}[\psi, \widetilde{\psi + \delta\psi}]$, where $P_{1,1}[\psi, \psi] = 0$ for identically prepared photons (coalescence), as it trivially follows from Eq. (1) and normalization $\langle \psi | \psi \rangle = 1$. A straightforward calculation from Eq. (1) yields

$$\Delta P_{1,1}[\psi] = \frac{1}{2} \frac{\Delta^2 (1 - |\alpha|^2)}{1 + \Delta(\alpha + \alpha^*) + \Delta^2}, \quad (2)$$

where $\Delta^2 \equiv (\delta\psi, \delta\psi)$, $\alpha \equiv (\psi, \delta\psi)/\Delta$ and $(\psi, \psi) = 1$. This result is *exact* and perfectly general, we have not made hypotheses on either the magnitude or the kind of the functional variation $\delta\psi$. In fact, we have solely used the basic properties of the scalar product in a complex linear space \mathcal{L} .

Equation (2) may be further developed by assuming that the functional deviation $\delta\psi$ is generated by the variation of a single DOF, represented by the *real* parameter f , in such a way that $\psi^A = \psi(f)$ and $\psi^B = \psi(f + \delta f)$, with $|\delta f| \ll |f|$. For example, if $f = \nu$, the photon at input port A has central frequency $\nu^A = \nu$ and the one entering port B has central frequency $\nu^B = \nu + \delta\nu$. Defining $\delta\psi \equiv \psi(f + \delta f) - \psi(f)$ permits to express ψ^B as above: $\psi^B = (\psi + \delta\psi) / \|\psi + \delta\psi\| \equiv \widetilde{\psi + \delta\psi}$. Formally expanding $\psi(f + \delta f)$ in powers of δf as [20]

$$\psi(f + \delta f) \simeq \psi(f) + \psi'(f) \delta f + \psi''(f) \delta f^2 / 2 + \dots, \quad (3)$$

with $\psi'(f) = \partial\psi(f)/\partial f$, $\psi''(f) = \partial^2\psi(f)/\partial f^2$, et cetera, we can straightforwardly obtain

$$\Delta^2 = \delta f^2 [(\psi', \psi') + \text{Re}(\psi'', \psi') \delta f + O(\delta f^2)], \quad (4)$$

and $\alpha = [(\psi, \psi') + (\psi, \psi'') \delta f / 2 + O(\delta f^2)] / \Delta$, where $\text{Re}(z) = (z + z^*)/2$, $\forall z \in \mathbb{C}$. Since from Eqs. (2) and (4) it follows that $\Delta P_{1,1}[\psi] \propto \Delta^2 = O(\delta f^2)$, we define the *rate of distinguishability* $R_f[\psi]$ of the photons with respect to the degree of freedom f via the relation

$$\begin{aligned} R_f^2[\psi] &\equiv \left. \frac{\partial^2 P_{1,1}[\psi(f), \psi(f + \delta f)]}{\partial f^2} \right|_{\delta f=0} \\ &= (\psi', \psi') - (\psi, \psi')^2, \end{aligned} \quad (5)$$

where $0 \leq R_f[\psi] \leq \|\psi'\|$, and $(\psi, \psi')^2 \leq 0$ because $0 = \partial(\psi, \psi)/\partial f = (\psi, \psi') + (\psi', \psi)$. The dimensionless parameter $R_f[\psi]\delta f$ has a straightforward physical meaning: it tell us how rapidly two identically prepared photons become distinguishable when we slightly vary, from $\psi(f)$ to $\psi(f + \delta f)$, the state of one photon with

respect to the other. The introduction of $R_f[\psi]$ has some remarkable consequences. Given a pair of photons prepared in the same state ψ , one can assert that they are *maximally indistinguishable* with respect to f if $R_f[\psi] \leq R_{f'}[\psi]$, $\forall f' \neq f$. In a complementary manner, provided two distinct pairs of photons, the first two photons prepared in the state ψ and the second ones in the state ϕ , one can say that the photons in the first pair are maximally indistinguishable with respect to ψ for a fixed f , if $R_f[\psi] \leq R_f[\phi]$, $\forall \phi \neq \psi$. In summary: “all photons are equal but some photons are more equal than others” [21], and $R_f[\psi]$ quantifies the degree of equality.

This result conclude the first part of this Letter. In the remainder, we will apply Eqs. (3-4) to the realistic case of optical Gaussian wave packets with well-defined spectral, spatiotemporal and polarization DOFs.

Gaussian wave packets.—Consider a single-photon wave packet of the form:

$$|\psi\rangle = \sum_{s=1}^2 \int d^3k \psi_s(\mathbf{k}) \hat{a}_s^\dagger(\mathbf{k}) |0\rangle, \quad (6)$$

where $|0\rangle$ denotes the ground state of the continuous Fock space and $\hat{a}_s(\mathbf{k})$ is the operator that annihilates one photon from the plane wave mode $\mathbf{e}_s(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{r})$, with $[\hat{a}_s(\mathbf{k}), \hat{a}_{s'}^\dagger(\mathbf{k}')] = \delta^{(3)}(\mathbf{k} - \mathbf{k}') \delta_{ss'}$ [22]. Here $\{\mathbf{e}_1(\mathbf{k}), \mathbf{e}_2(\mathbf{k}), \mathbf{k}/|\mathbf{k}|\}$ denotes a right-handed orthonormal basis set attached to the wave vector \mathbf{k} . The normalization of the state is ensured by requiring

$$\langle \psi | \psi \rangle = \sum_{s=1}^2 \int d^3k |\psi_s(\mathbf{k})|^2 = 1. \quad (7)$$

The spectral amplitude $\psi_s(\mathbf{k})$ determines the shape and the polarization of the beam. It may be deduced by imposing the quantum-classical correspondence

$$\langle 0 | \hat{\mathbf{E}}^{(+)}(\mathbf{r}, t) | \psi \rangle = \mathbf{E}_{cl}^{(+)}(\mathbf{r}, t), \quad (8)$$

where $\mathbf{E}_{cl}^{(+)}(\mathbf{r}, t)$ is the positive-frequency part of the classical field wave packet whose energy equals the mean energy of the photon in the state $|\psi\rangle$, and

$$\begin{aligned} \hat{\mathbf{E}}^{(+)}(\mathbf{r}, t) &= \frac{i}{(2\pi)^{3/2} \epsilon_0^{1/2}} \sum_{s=1}^2 \int d^3k \sqrt{\frac{\hbar\omega}{2}} \hat{a}_s(\mathbf{k}) \mathbf{e}_s(\mathbf{k}) \\ &\quad \times \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)], \end{aligned} \quad (9)$$

with $\omega = c|\mathbf{k}| \equiv ck$, c being the speed of light in vacuum and ϵ_0 the vacuum permittivity. The expression for $\mathbf{E}_{cl}^{(+)}(\mathbf{r}, t)$ is given by the right side of Eq. (9) with the quantum operator $\hat{a}_s(\mathbf{k})$ replaced by the classical amplitude $\tilde{a}_s(\mathbf{k})$. Then, by substituting from Eqs. (6) and (9) into Eq. (8), one obtains $\psi_s(\mathbf{k}) = \tilde{a}_s(\mathbf{k})$. The total energy contained in such wave packet is given by $\mathcal{E} = \int d^3k \hbar\omega (|\tilde{a}_1(\mathbf{k})|^2 + |\tilde{a}_2(\mathbf{k})|^2)$.

Without loss of generality, we assume that $\tilde{a}_s(\mathbf{k}) = \varepsilon_s(\mathbf{k})E(\mathbf{k})$, where $E(\mathbf{k})$ and $\varepsilon_s(\mathbf{k})$ are the scalar and the vector spectral amplitudes of the field, respectively. $E(\mathbf{k})$ determines the spatial characteristics of the field, and $\varepsilon_s(\mathbf{k})$ the polarization. Here, we consider a collimated, quasi-monochromatic wave packet, with central wave vector \mathbf{k}_0 and central frequency $\omega_0 = c|\mathbf{k}_0| \equiv ck_0$. We choose a normalized Gaussian spectral amplitude $E(\mathbf{k}) = \gamma(\mathbf{k} - \mathbf{k}_0)$, where

$$\gamma(\mathbf{q}) = \frac{(\det V)^{1/4}}{\pi^{3/4}} \exp \left[-i(\mathbf{q}, \mathbf{r}_0) - \frac{1}{2}(\mathbf{q}, V\mathbf{q}) \right], \quad (10)$$

with $V^{-1} = \text{diag}(\sigma_1^2, \sigma_2^2, \sigma_3^2)$. This choice for V yields a factorizable spectral amplitude $\gamma(\mathbf{q}) = g(q_1)g(q_2)g(q_3)$ with $g(q_n) = \exp[-iq_n r_{0n} - q_n^2/(2\sigma_n^2)]/(\pi^{1/4}\sqrt{\sigma_n})$. Clearly, it is possible to consider a more general positive definite symmetric matrix V that couples different wave vector coordinates. We will see later that such a coupling may have dramatic consequences upon the rate of distinguishability of photons. In Eq. (10) the real vector $\mathbf{r}_0 = \{r_{01}, r_{02}, r_{03}\}$ gives the position, at time $t = 0$, of the center of the wave packet. We fix $\varepsilon_s(\mathbf{k})$ assuming that the wave packet has passed across a polarizer that selects a *uniform* field polarization parallel to $\mathbf{p} \in \mathbb{C}^3$ and perpendicular to \mathbf{k}_0 , with $|\mathbf{p}|^2 = 1$ and $(\mathbf{k}_0, \mathbf{p}) = 0$. In this case, it becomes natural to define $\varepsilon_s(\mathbf{k})$ as the normalized projection of \mathbf{p} upon $\mathbf{e}_s(\mathbf{k})$, namely [23, 24]: $\varepsilon_s(\mathbf{k}) = (\mathbf{e}_s(\mathbf{k}), \mathbf{p})/\sqrt{1 - |(\mathbf{p}, \mathbf{k})|^2/k^2}$, with $|\varepsilon_1(\mathbf{k})|^2 + |\varepsilon_2(\mathbf{k})|^2 = 1$ by definition.

The Gaussian distribution $\gamma(\mathbf{k} - \mathbf{k}_0)$ implies that the wave packet is concentrated in a region of the \mathbf{k} -space of “volume” $\sigma_1\sigma_2\sigma_3$ centered at \mathbf{k}_0 . Then, the assumptions of collimation and quasi-monochromaticity entail the constraints $\sigma_i \ll k_0$, ($i = 1, 2, 3$). In this case, the total energy of the wave packet can be written as $\mathcal{E} = \int d^3k \hbar\omega |\gamma(\mathbf{k} - \mathbf{k}_0)|^2 \simeq \hbar\omega_0$, where $\int d^3k |\gamma(\mathbf{k} - \mathbf{k}_0)|^2 = 1$ by definition.

For quasi-monochromatic and collimated beams the spectral amplitude $\psi_s(\mathbf{k}) = \varepsilon_s(\mathbf{k})\gamma(\mathbf{k} - \mathbf{k}_0)$ contains $(3 + 3) + 3 + 3 = 12$ independent real parameters corresponding to the (spectral) \oplus spatial \oplus polarization DOFs: $(\mathbf{k}_0 \oplus \{\sigma_1, \sigma_2, \sigma_3\}) \oplus \mathbf{r}_0 \oplus \{\mathbf{p} \in \mathbb{C}^3 : |\mathbf{p}|^2 = 1 \wedge (\mathbf{k}_0, \mathbf{p}) = 0\}$. Note that the central frequency ω_0 is *not* an additional independent parameter, since $\omega_0 = c|\mathbf{k}_0|$. Thus, as we claimed in the Introduction, a single photon is not a so “elementary” particle as it is usually believed. In fact, we have at our disposal as many 12 (actually 15 if we consider a non-diagonal symmetric V), independent parameters to characterize the quantum state of the photon. Each of these parameters can be taken as the variable f to evaluate the rate of distinguishability $R_f[\psi]$. This calculation will be the goal of the remainder.

Rate of distinguishability.—Using rather standard methods of calculation [25, 26], it is not difficult to show that the coincidence probability (1) can be expressed in

terms of the spectral amplitudes $\psi_s^A(\mathbf{k})$ and $\psi_s^B(\mathbf{k})$ of the input photons as

$$P_{1,1}[\psi^A, \psi^B] = \frac{1}{2} \left[1 - \left| \sum_{s=1}^2 \int d^3k \psi_s^A(\mathbf{k}) \psi_s^{B*}(\underline{\mathbf{k}}) \right|^2 \right], \quad (11)$$

where $\underline{\mathbf{k}}$ has components $\{-k_1, k_2, k_3\}$. This change of sign in the 1-coordinate is due to the parity inversion occurring by reflection at the BS. Hereafter, we assume $\psi_s^A(\mathbf{k}) = \psi_s(\mathbf{k}, f)$ and $\psi_s^B(\mathbf{k}) = \psi_s(\underline{\mathbf{k}}, f + \delta f)$. Moreover, for concreteness, we choose the 3-axis of the Cartesian reference frame directed along \mathbf{k}_0 , namely $\mathbf{k}_0 = \{0, 0, k_0\}$.

The explicit values of R_f , calculated from Eq. (5), are given in Table I below, for spectral and spatial DOFs:

f	k_{0n}	σ_n	r_{0n}
$\sqrt{2} R_f$	$\frac{1}{\sigma_n}$	$\frac{1}{\sigma_n}$	σ_n

TABLE I. Rate of distinguishability for several spectral and spatial degrees of freedom of the photons, with $n = 1, 2, 3$.

A remarkable consequence from Table I, is that for the complementary position/wave-vector variables, the following Fourier-transform equality holds: [27]

$$R_{k_{0n}} R_{r_{0n}} = 1/2, \quad \forall n = 1, 2, 3. \quad (12)$$

Table I furnishes some valuable information. Consider, for example, the last column: it shows that $R_{r_{0n}} \delta r_{0n}$ is equal to the ratio between the variation δr_{0n} and the $1/e$ width $\sqrt{2}/\sigma_n$ of the photon wave packet in configuration space. This is in agreement with intuition: imagine the cross-section of each photon wave packet as a disc of radius $\sqrt{2}/\sigma_n$. Starting from an initial condition of perfect superposition between the two discs, suppose to shift one disc with respect to the other by the amount δr_{0n} . Now, if $\delta r_{0n} \ll \sqrt{2}/\sigma_n$ the two discs have still a large superposition and the two photons remain largely indistinguishable. Vice versa, if $\delta r_{0n} \sim \sqrt{2}/\sigma_n$ the two discs separate completely and the superposition drops to zero. In this case the photons become “quickly” distinguishable. Analogous reasonings may be reproduced for the others DOFs.

Next, we consider the case of a non-factorable spectral amplitude, which couples wave vector coordinates 1 and 2. Equation (10) still holds, but now V has elements $V_{nm} = \delta_{nm}/\sigma_n^2 - (\delta_{n1}\delta_{m2} + \delta_{n2}\delta_{m1})/\sigma_{nm}$, where the real parameter σ_{12} establishes the coupling, with $\sigma_{12}^2 > \sigma_1^2\sigma_2^2$ as required by positive definiteness of V . A straightforward calculation furnishes

$$\sqrt{2} R_{\sigma_n} = \frac{1}{\sigma_n} \frac{1}{1 - \rho^2}, \quad (n = 1, 2), \quad (13)$$

where $\rho \equiv \sigma_1\sigma_2/\sigma_{12}$, with $|\rho| < 1$. If $\rho = 0$ (uncoupled DOFs) we recover the results of Table I. Vice

versa, for increasing coupling one has $\rho \rightarrow 1$ and R_{σ_n} grows unboundedly. This result is of particular relevance to experimentalists: it tells us that wave packets whose cross section has the shape of an ellipse whose either major or minor axis does not lay on the plane of incidence (see Fig. 1), are much more sensitive to mode-mismatch than cylindrically symmetric wave packets. This fact strongly degrades photon indistinguishability and should be avoided, for example, in coalescence experiments [9, 10].

Finally, we examine the polarization DOFs of the two photons. Let us parameterize \mathbf{p}^A and \mathbf{p}^B as $\mathbf{p}^\lambda = \{\cos \vartheta^\lambda \exp(i\varphi_1^\lambda), \sin \vartheta^\lambda \exp(i\varphi_2^\lambda), 0\}$, with $\lambda = A, B$ and $|\mathbf{p}^A| = |\mathbf{p}^B| = 1$. The results, as expansions in powers of σ_1, σ_2 , are

$$R_\vartheta^2 \simeq 1 + \frac{\sigma_1^2 - \sigma_2^2}{2k_0^2} \cos(2\vartheta) + \dots, \quad (14a)$$

$$R_{\varphi_n}^2 \simeq \frac{\sin^2(2\vartheta)}{4} \left[1 + \frac{\sigma_1^2 - \sigma_2^2}{2k_0^2} \cos(2\vartheta) + \dots \right], \quad (14b)$$

with $n = 1, 2$. Unlike in the spectral and scalar cases, R_ϑ and R_{φ_n} are dimensionless quantities. From a physical point of view, this means that there is not a natural scale for the variation of the polarization DOFs. From Eqs. (14) we see that for astigmatic wave packets, namely for $\sigma_1 \neq \sigma_2$, there is a coupling between spectral and polarization DOFs that affects both R_ϑ and R_{φ_n} . The term $(\sigma_1^2 - \sigma_2^2)/(2k_0^2)$ may be interpreted as a manifestation of the *unavoidable* spin-orbit coupling occurring in transverse electromagnetic fields [28]. In addition, its absolute value furnishes the visibility of the coincidence fringes [29]. Equation (14b) shows that $R_{\varphi_n} \propto \sin(2\vartheta)$. As a consequence, for linearly polarized states with $2\vartheta = 0, \pm\pi, \pm 2\pi, \dots$, the phase is not a relevant DOF and $R_{\varphi_n} = 0$.

Conclusions.— In this Letter we have studied the issue of particle indistinguishability from an operational point of view. In this way, we could circumvent most of the pitfalls usually encountered in this subtle topic [1–5]. In practice, from the analysis of a standard two-photon interference experiment we could introduce a useful parameter, the rate of distinguishability, which furnishes a quantitative measure of the indistinguishability of photons prepared in a given state. Being operationally defined, this parameter is susceptible of experimental investigation via the measurement of the two-photon coincidence probability.

Our main results are summarized by Eqs. (2,5,13), and Table I. In particular, Eq. (13) quantifies the degradation of photon distinguishability due to coupling between different DOFs. As a final remark, we stress that we have defined the rate of distinguishability considering the variation of a single DOF of the photons. However, one could also consider simultaneous variations of two or more DOFs and introduce the corresponding *cross-rates*

of distinguishability with respect to f, f', \dots

* falk.toeppel@mpl.mpg.de

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